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Fractional Super-Multi-Virasoro Algebra

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An n-dimensional fractional supersymmetry theory is algebraically constructed on the n-dimensional multicomplex space $M\mathbb{C}_n$. By emphasizing its appearance as a special case of generalized Clifford algebra (GCA), we formulate the fractional superspace $FM\mathbb{C}_n$ through a generalized Grassmann algebra (GGA) and construct the generators and the covariant derivative of FSUSY on $FM\mathbb{C}_n$. The generators of FSUSY are extended to get n copies of the fractional centerless super-Virasoro algebra.

INTRODUCTION

Among the possible algebras, those induced by bilinear relations have a special status, probably due to the bilinear aspect of fundamental objects such as quadratic metrics, commutators, anticommutators, etc. Algebras going beyond the quadratic ones were constructed in the 1970s from underlying polynomials of degree higher than two. They are dubbed Clifford algebras of polynomials and n-exterior algebras [1, 2] by mathematicians. The matricial representation of such algebras [3] leads to a natural algebraic extension of the Clifford and Grassmann algebra [4, 5]. The two resulting basic structures, generalized Clifford and Grassmann algebras (GCA and GGA), endow a differential structure on noncommutative variables which allow one to build a theory beyond supersymmetry [10].

These families endow also, from a mathematical point of view, natural extensions of quadratic theories (complex numbers, quaternions, etc.). Multi-complex numbers $M\mathbb{C}_n$ [6, 7] were introduced in analogy to complex numbers with respect to the usual Clifford algebra.

Multicomplex algebra $M\mathbb{C}_n$ is an n-dimensional **R**-algebra and has a much richer structure than the field of ordinary complex algebra. It has been

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used to extend powerful tools of two-dimensional (super)-conformal field theories to higher dimension [8, 9] on $M\mathbb{C}_n$ under the assumption $n = 2^p$.

Because of the $\mathbb{Z}_n$-graded structure of generalized Clifford (Grassmann) algebra, and due to the fact that the generators are in a representation of the tress group, they play a considerable role in the study of deformations of certain algebras, quantum groups, and the extension of supersymmetric theories known as fractional supersymmetry theory (FSUSY). FSUSY has been considered in one dimension [10–12], where this symmetry can be seen as an Fth root of the time translation ∂_t; or in two dimensions as an Fth root of conformal transformations [13]. $F = 2$ corresponds to the usual supersymmetry. The case of 1D fractional supersymmetry leads to a new equation acting on the states which are in the representation of the braid group [12]; the method adopted there is similar to the one leading to the Dirac equation in one dimension using supersymmetry [14]. The 2D fractional supersymmetry theory has been algebraically constructed and the Lagrangian derived using an adapted superspace [13] via a heterotic extension of the complex plane by the help of generalized Grassmann variables and its differential structure. Three-dimensional generalizations of fractional supersymmetry generate symmetries which connect fractional spin states or anyons [14].

A very interesting interpretation of FSUSY was given in one dimension as an appropriate limit of the braided line [17]; 2D FSUSY was used to describe new 2D integrable models, and 3D FSUSY led to generalizations of the well-known Wess–Zumino model [18]. If particular dimensions can reveal interesting behavior, it should be worthwhile to study n-dimensional ($n \geq 4$) FSUSY extensions to understand the consequences of such extensions in relation to n-dimensional physics.

The aim of the present paper is to show that the multicomplex space $M\mathbb{C}_n$, which is in fact a GCA generated by a canonical generator e, fulfilling the basic relation [6]

$$e^n = -1, \qquad n \in N^*$$

provides the possibility to extend almost of the results of Fleury *et al.* [13] and those on the supermulticomplex space $SM\mathbb{C}_n$ [8] via a realization of the fractional supersymmetry FSUSY on the fractional superspace of $M\mathbb{C}_n$ denoted $FM\mathbb{C}_n$. Using generalized Grassmann variables, we will formulate $FM\mathbb{C}_n$ and construct the generators and the covariant derivatives of FSUSY on $FM\mathbb{C}_n$; they are extended and lead to n copies of fractional centerless super-Virasoro (FSV) algebra in ref. [19]; the main reason for this difference is that ours closes through local (but nonquadratic) relations, whereas FSV closes with nonlocal (but quadratic) ones.

1. PRELIMINARIES

A property not often emphasized of the set of complex numbers is its appearance as a special case of Clifford algebra. All these algebras have in common their definition from quadratic or bilinear relations and consequently admit a $\mathbb{Z}_2$-graded structure. However, mathematicians have obtained, in the spirit of usual Clifford algebras, new algebras defined from n-linear relations and leading to an underlying $\mathbb{Z}_n$-graded structure. The so-called polynomial Clifford algebras can be defined, à la Dirac, through linearization of a homogeneous polynomial P,

$$P(x) = \sum_{i_j=0}^{k} x_{i_1} \cdots x_{i_n} g_{i_1 \ldots i_n} \tag{1.1}$$

of degree n and k variables, it is generated by $g_1, \ldots, g_k$ [1], which satisfy

$$P(x_1, \ldots, x_n) = (x_1 g_1 + \cdots + x_k g_k)^n \tag{1.2}$$

$$\{g_{i_1}, g_{i_2}, \ldots, g_{i_n}\} = g_{i_1 \ldots i_n} \tag{1.3}$$

where the bracket is defined as

$$\{g_{i_1}, g_{i_2}, \ldots, g_{i_n}\} = \frac{1}{n!} \sum_{\text{perm}\sigma} g_{i_{\sigma(1)}} g_{i_{\sigma(2)}} \cdots g_{i_{\sigma(n)}} \tag{1.4}$$

From this linearization the so-called generalized Clifford algebra (GCA) emerges quite naturally [3–5].

The GCA denoted C_n^r is generated by a set of r canonical generators $e_1, \ldots, e_r$ satisfying

$$e_i e_j = \omega^{\mathrm{Sg}(j-i)} e_j e_i, \qquad e_i^n = -1, \qquad i, j = 1, \ldots, r \tag{1.5}$$

where $\omega = \exp(2i\pi/n)$ is an nth primitive root of unity and $\mathrm{Sg}(\cdot)$ is the usual sign function; the GCA and its associated generalized Grassmann algebra [3, 5] (with $e_i^n = 0$) have a wide range of applications in physics [15]. C_n^r is defined as a $\mathbb{C}$-algebra, but, when $r = 1$, C_n^1 can be defined as an $\mathbf{R}$-algebra. Looking at Eq. (1.5), we see that the case of one generator can be put on the same footing as the complex numbers with respect to the Clifford algebra. This algebra was dubbed multicomplex numbers, and efforts in its development [6] are motivated by quantum mechanics. These ideas are based on homogeneous forms of degree higher than two [20].

The set of multicomplex numbers is generated by one element e [6] such that

$$M\mathbb{C}_n = \left\{ z = \sum_{i=0}^{n-1} x_i e^i,\ e^n = -1/n \in N^*,\ x_i \in \mathbf{R} \right\} \tag{1.6}$$

it was shown that most of the theorems of complex analysis can be extended to the $M\mathbb{C}_n$, $n \geq 2$, spaces [7]. Among the $M\mathbb{C}_n$ spaces, the one for which $n = 2^p$ has a special status [7]; in that case, one defines the $\mathfrak{p}$th conjugate of a multicomplex number z as follows:

$$\overset{(\mathfrak{p})}{z} = \sum_{i=0}^{n-1} x_i e^{i(2\mathfrak{p}+1)}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1 \tag{1.7}$$

This conjugation satisfies

$$\overset{(\mathfrak{p}+n)}{z} = \overset{(\mathfrak{p})}{z}, \qquad \overset{(\mathfrak{p})}{z}_1 \overset{(\mathfrak{p})}{z}_2 = \overset{(\mathfrak{p})}{\widehat{z_1 z_2}}$$

It was established that the set of $M\mathbb{C}_n$-numbers can be equipped with the pseudo norm

$$\|z\|^n = \prod_{\mathfrak{p}=0}^{n-1} \overset{(\mathfrak{p})}{z} \tag{1.8}$$

by means of the product of n elements of $M\mathbb{C}_n$. The notion of the $\mathfrak{p}$th conjugate allows us to see any element z of $M\mathbb{C}_n$ as parametrized by the n multicomplex numbers $\overset{(\mathfrak{p})}{z}$. This is equivalent to saying that Eq. (1.7) can be inverted. As an easy example, for $n = 2$, a complex number $z = x_0 + x_1 i$ parametrized by x_0 and x_1 can also be parametrized by the two conjugates $\overset{(0)}{z} = z$ and $\overset{(1)}{z} = \bar{z}$.

It is then convenient to introduce the differential operators

$$\overset{(\mathfrak{p})}{\partial} = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-i\mathfrak{p}} e^{-i} \frac{\partial}{\partial x_i} \tag{1.9}$$

satisfying $\overset{(\mathfrak{p})}{\partial} \overset{(k)}{z} = \delta^{k\mathfrak{p}}$.

Moreover, it is worth stressing that most of the results of usual complex number analysis remain true for $M\mathbb{C}_n$ numbers, whether for algebraic [6] or analytic properties [7].

A main property useful for the sequal is that a mapping F: $M\mathbb{C}_n \to M\mathbb{C}_n$ is differentiable at z iff

$$\overset{(\mathfrak{p})}{\partial} \overset{(k)}{F} = 0 \qquad \forall \mathfrak{p} \neq k \tag{1.10}$$

A function satisfying Eq. (1.10) will be called holomorphic. We recall some of the properties of the $M\mathbb{C}_n$ numbers and refer to refs. 6 and 7 for more details.

2. *n*-DIMENSIONAL FSUSY ON THE FRACTIONAL SUPERSPACE $FM\mathbb{C}_n$

By analogy with the building of the fractional superspace of $\mathbb{C}$ and as for heterotic strings [16], where z and $\bar{z}$ are extended differently [$z \to (z, \theta)$

and $\bar{z}$ remains unaffected], to define the fractional superspace of $M\mathbb{C}_n$, to $\overset{(0)}{z}, \overset{(1)}{z}, \ldots, \overset{(n-1)}{z}$, we associate n real generalized Grassmann variables $\overset{(0)}{\theta}, \overset{(1)}{\theta}, \ldots, \overset{(n-1)}{\theta}$. In other words, a point z in $M\mathbb{C}_n$ is completely determined by its $n\mathfrak{p}$th conjugate $\overset{(\mathfrak{p})}{z}$ as seen before; thus $FM\mathbb{C}_n$ is the set of points parametrized by the multicomplex numbers $\overset{(\mathfrak{p})}{z}$ and the real generalized Grassmann variables $\overset{(\mathfrak{p})}{\theta}$, $\mathfrak{p} = 0, 1, 2, \ldots, n-1$; the construction acts separately on the 1-sectors.

We need first to recall briefly the underlying algebra which allows us to define FSUSY on $M\mathbb{C}_n$. Generalized Grassmann variables are denoted $\overset{(\mathfrak{p})}{\theta}$, $\mathfrak{p} = 0, 1, 2, \ldots, n-1$, and the associated derivatives $\partial_{(\mathfrak{p})}$ and $\delta_{(\mathfrak{p})}$ satisfy the basic algebraic relations

$$\begin{aligned} \partial_{(\mathfrak{p})}\overset{(\mathfrak{p})}{\theta} - q\overset{(\mathfrak{p})}{\theta}\partial_{(\mathfrak{p})} &= 1, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1 \\ \delta_{(\mathfrak{p})}\overset{(\mathfrak{p})}{\theta} - q^{-1}\overset{(\mathfrak{p})}{\theta}\delta_{(\mathfrak{p})} &= 1, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1 \\ \overset{(\mathfrak{p})}{\theta}{}^{F} = 0; \qquad \partial_{(\mathfrak{p})}{}^{F} = \delta_{(\mathfrak{p})}{}^{F} &= 0, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1 \\ \partial_{(\mathfrak{p})}\delta_{(\mathfrak{p})} = q^{-1}\delta_{(\mathfrak{p})}\partial_{(\mathfrak{p})}&, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1 \end{aligned} \tag{2.1}$$

where q is a primitive Fth root of the unity; it can be chosen as $q = \exp(2i\pi/F)$. A consequence of relations (2.1) is a Leibnitz rule leading to

$$\partial_{(\mathfrak{p})}(\overset{(\mathfrak{p})}{\theta}{}^{a}) = \{a\}\overset{(\mathfrak{p})}{\theta}{}^{a-1} \tag{2.2}$$

where $\{a\} = (1 - q^a)/(1 - q)$ (with the derivative $\delta_{(\mathfrak{p})}$ we would have obtained the same result with the substitution $q \to q^{-1}$).

The relations which mix the $\mathfrak{p}$-sectors are

$$\begin{aligned} \overset{(\mathfrak{p})}{\theta}\overset{(k)}{\theta} &= q\overset{(k)}{\theta}\overset{(\mathfrak{p})}{\theta} \\ \partial_{(\mathfrak{p})}\partial_{(k)} &= q\partial_{(k)}\partial_{(\mathfrak{p})} \\ \partial_{(\mathfrak{p})}\overset{(k)}{\theta} &= q^{-1}\overset{(k)}{\theta}\partial_{(\mathfrak{p})} \\ \partial_{(k)}\overset{(\mathfrak{p})}{\theta} &= q\overset{(\mathfrak{p})}{\theta}\partial_{(k)} \\ \delta_{(\mathfrak{p})}\delta_{(k)} &= q\delta_{(k)}\delta_{(\mathfrak{p})} \\ \delta_{(\mathfrak{p})}\overset{(k)}{\theta} &= q^{-1}\overset{(k)}{\theta}\delta_{(\mathfrak{p})} \end{aligned} \tag{2.3a}$$

$$\delta_{(k)}\overset{(\mathfrak{p})}{\theta} = q\overset{(\mathfrak{p})}{\theta}\delta_{(k)} \tag{2.3b}$$

where $\mathfrak{p} < k$; $\mathfrak{p}, k = 0, 1, 2, \ldots, n-1$.

The algebra defined in Eqs. (2.1) and (2.3) is stable neither under complex conjugation nor under the permutation of θ indices (we denote this permutation by σ). However, it is stable under the composition of both

$$(AB)^{*\circ\sigma} = A^{*\circ\sigma}B^{*\circ\sigma}$$

which defines an automorphism of the algebra exchanging $(\overset{(k)}{z}, \overset{(k)}{\theta})$ and $(\overset{(\mathfrak{p})}{z}, \overset{(\mathfrak{p})}{\theta})$, $\mathfrak{p}, k = 0, 1, 2, \ldots, n-1$. With such an automorphism, $(\overset{(\mathfrak{p})}{z}, \overset{(\mathfrak{p})}{\theta}, \partial_{(\mathfrak{p})}, \delta_{(\mathfrak{p})})$ is mapped onto $(\overset{(k)}{z}, \overset{(k)}{\theta}, \partial_{(k)}, \delta_{(k)})$ and vice versa, so we see that we have a connection between the $\mathfrak{p}$-sectors, $\mathfrak{p} = 0, 1, 2, \ldots, n-1$. In fact, $* \circ \sigma$ has the same significance as the one related to FSUSY on $\mathbb{C}$.

We can remark that

$$\delta_{(\mathfrak{p})} = (\partial_{(\mathfrak{p})})^*, \qquad (\partial_{(\mathfrak{p})}\overset{(\mathfrak{p})}{\theta})^* = \overset{(\mathfrak{p})}{\theta}\delta_{(\mathfrak{p})}$$

So, we refer to the fractional superspace $FM\mathbb{C}_n$ by

$$(\overset{(\mathfrak{p})}{z}, \overset{(\mathfrak{p})}{\theta}, \partial_{(\mathfrak{p})}, \delta_{(\mathfrak{p})})$$

3. THE FRACTIONAL SUPER-MULTIVIRASORO ALGEBRA

From the above algebra, we can build the generators and the covariant derivatives of FSUSY on $FM\mathbb{C}_n$ associated to the $\overset{(\mathfrak{p})}{z}$-modes, $\mathfrak{p} = 0, 1, 2, \ldots, n-1$:

$$\overset{(\mathfrak{p})}{Q} = \delta_{(\mathfrak{p})} + \frac{(1-q^{-1})^{F-1}}{F}\overset{(\mathfrak{p})}{\theta}{}^{F-1}\overset{(\mathfrak{p})}{\partial}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1 \tag{3.1}$$

$$\overset{(\mathfrak{p})}{D} = \partial_{(\mathfrak{p})} + \frac{(1-q)^{F-1}}{F}\overset{(\mathfrak{p})}{\theta}{}^{F-1}\overset{(\mathfrak{p})}{\partial}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1$$

which satisfy

$$\overset{(\mathfrak{p})}{Q}{}^F = \overset{(\mathfrak{p})}{D}{}^F = \overset{(\mathfrak{p})}{\partial}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1 \tag{3.2}$$

$$\overset{(\mathfrak{p})}{Q}\overset{(\mathfrak{p})}{D} = q\overset{(\mathfrak{p})}{D}\overset{(\mathfrak{p})}{Q}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1$$

and where

$$\overset{(\mathfrak{p})}{\partial} = \partial_{\overset{(\mathfrak{p})}{z}}; \qquad \partial_{(\mathfrak{p})} = \partial_{\overset{(\mathfrak{p})}{\theta}}$$

Relations (3.2) can be obtained directly using Eqs. (2.1) and (2.3). The generators (3.1) can be extended:

$$\begin{aligned}
\overset{(\mathfrak{p})}{L}_n &= \overset{(\mathfrak{p})}{z}{}^{1-n}\overset{(\mathfrak{p})}{\partial} - \frac{1}{F}(n-1)\overset{(\mathfrak{p})}{z}{}^{-n}\overset{(\mathfrak{p})}{\mathcal{N}}, \qquad n \in \mathbb{Z} \\
\overset{(\mathfrak{p})}{G}_L &= \overset{(\mathfrak{p})}{z}{}^{1/F-r}\left(\partial_{(\mathfrak{p})} + \frac{(1-q)^{F-1}}{F}\overset{(\mathfrak{p})}{\theta}{}^{F-1}\overset{(\mathfrak{p})}{\partial}\right) \qquad (3.3)\\
&\quad - \frac{(1-q)^{F-1}}{F}\left(r - \frac{1}{F}\right)\overset{(\mathfrak{p})}{z}{}^{1/F-r-1}\overset{(\mathfrak{p})}{\theta}{}^{F-1}\overset{(\mathfrak{p})}{\mathcal{N}}, \qquad r \in \mathbb{Z} + \frac{1}{F}
\end{aligned}$$

and lead to n copies of fractional super-Virasoro algebra without central extension:

$$\begin{aligned}
[\overset{(\mathfrak{p})}{L}_n, \overset{(k)}{L}_m] &= (n-m)\overset{(\mathfrak{p})}{L}_{m+n}\delta_{\mathfrak{p},k} \\
[\overset{(\mathfrak{p})}{L}_n, \overset{(k)}{G}_r] &= \left(\frac{n}{F} - r\right)\overset{(\mathfrak{p})}{G}_{n+r}\delta_{\mathfrak{p},k} \qquad (3.4)\\
\{\overset{(\mathfrak{p})}{G}_{r_1}, \ldots, \overset{(\mathfrak{p})}{G}_{r_F}\} &= \overset{(\mathfrak{p})}{L}_{r_1+\cdots+r_F}
\end{aligned}$$

where $\delta_{\mathfrak{p},k}$ is the usual Kronecker symbol, $\overset{(\mathfrak{p})}{\mathcal{N}}$ is given by

$$\overset{(\mathfrak{p})}{\mathcal{N}} = \sum_{i=1}^{n-1}\frac{(1-q)^i}{(1-q^i)}\overset{(\mathfrak{p})}{\theta}{}^{i}\partial_{(\mathfrak{p})}{}^{i}$$

and $\{\cdots\}$ is the multilinear symmetric product defined by Eq. (1.4).

4. FRACTIONAL SUPERFIELD ON $FM\mathbb{C}_n$

In the fractional superspace $FM\mathbb{C}_n$, a fractional superfield decomposes as

$$\begin{aligned}
&\phi(\overset{(0)}{z}, \overset{(1)}{z}, \ldots, \overset{(n-1)}{z}, \overset{(0)}{\theta}, \overset{(1)}{\theta}, \ldots, \overset{(n-1)}{\theta}) \\
&\quad \sim \sum_{i_0=0}^{n-1}\cdots\sum_{i_{n-1}=0}^{n-1}\overset{(0)}{\theta}{}^{i_0}\cdots\overset{(1)}{\theta}{}^{i_{n-1}}\psi_{i_0/F,\ldots,i_{n-1}/F}(\overset{(0)}{z}, \overset{(1)}{z}, \ldots, \overset{(n-1)}{z}) \qquad (4.1)
\end{aligned}$$

we note that for $n = 2$, this decomposition reduces to the one on the complex fractional superspace, $(z, \bar{z}, \theta, \bar{\theta})$. We have many kinds of fields:

$$\psi_{i_0/F,\ldots,i_{\mathfrak{p}}/F=0,\ldots,i_{n-1}/F}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1$$
$$\psi_{i_0/F,\ldots,i_{\mathfrak{p}}/F=0,\ldots,i_{\mathfrak{p}}/F=0,\ldots,i_{n-1}/F}, \qquad \mathfrak{p}, r = 0, 1, 2, \ldots, n-1 \tag{4.2}$$
$$\psi_{0,\ldots,0,i_{\mathfrak{p}}/F,0,\ldots,0}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1$$

The various components of ϕ have nontrivial $\mathbb{Z}_F$-graduation, and are of grade $i_0, i_1, \ldots, i_{n-1}$; they satisfy, because of their grad, the q-mutation relations

$$(\psi_{i_0/F,\ldots,i_{n-1}/F})^F = 0 \tag{4.3}$$
$$\overset{(\mathfrak{p})}{\theta}\psi_{i_0/F,\ldots,i_{n-1}/F} = q^{-(i_0+i_1+\cdots+i_{n-1})}\psi_{i_0/F,\ldots,i_{n-1}/F}\overset{(\mathfrak{p})}{\theta}, \qquad \mathfrak{p} = 0, 1, 2, \ldots, n-1$$

We note that the covariant derivative $\overset{(\mathfrak{p})}{D}$ commutes with the FSUSY transformation $\overset{(\mathfrak{p})}{\varepsilon}\overset{(\mathfrak{p})}{Q}$:

$$\delta\overset{(\mathfrak{p})}{D}\phi = \overset{(\mathfrak{p})}{Q}\delta\phi$$

where $\overset{(\mathfrak{p})}{\varepsilon}$ is the parameter of the FSUSY transformation; its q-mutations with $\psi_{i_0/F,\ldots,i_{n-1}/F}$ are identical to those of $\overset{(\mathfrak{p})}{\theta}$ with $\psi_{i_0/F,\ldots,i_{n-1}/F}$.

The transformations of the superfield are

$$\delta\phi = \overset{(\mathfrak{p})}{\varepsilon}\overset{(\mathfrak{p})}{Q}\phi$$

The relation $\overset{(\mathfrak{p})}{\varepsilon}\overset{(\mathfrak{p})}{\theta} = q^{-1}\overset{(\mathfrak{p})}{\theta}\overset{(\mathfrak{p})}{\varepsilon}$ ensures that $\overset{(\mathfrak{p})}{D}$ is a covariant derivative, as it should be in order to build the FSUSY invariant action [13]. The algebraic structure developed here will be useful for constructing the fractional model on $FM\mathbb{C}_n$; this will be analyzed elsewhere.

Finally, note that we have seen that most of the properties of a 2D fractional supersymmetry theory on $\mathbb{C}$ can be extended to the multicomplex space $M\mathbb{C}_n$ via an algebraic method on $FM\mathbb{C}_n$ called “fractional super-multicomplex space. This could be useful for understanding the consequences of FSUSY extensions in relation to n-dimensional physics. Notice that this is the first step to be considered, as we did not discuss the representation theory of the FSUSY on $M\mathbb{C}_n$.

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